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PERTURBATION SOLUTIONS FOR MELTING OR FREEZING IN ANNULAR REGIONS INITIALLY NOT AT THE FUSION TEMPERATURE

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Abstract—This paper deals with the inward solidification of liquid in an annular space which is initially not at the fusion temperature. The outer cylindrical surface is maintained at a subfreezing temperature while the inner cylindrical boundary is assumed to be either insulated or maintained at constant temperature. New perturbation solutions are obtained for the temperature distribution and the interface motion. The perturbation parameter $\varepsilon = C_s(T_f - T_a)/L$ is the ratio of the sensible heat of the solid-phase to the latent heat of fusion. The non-uniformity of the long-time scale solutions is treated by constructing inner expansions in the short-time scale. The two solutions are matched using asymptotic theory.

The solutions for the insulated case do not depart markedly, except on the short time scale, from the corresponding solutions for liquids which are initially at the fusion temperature T_f. In contrast, the solutions for the isothermal inner boundary depart substantially from those with an initial temperature equal to T_f. This is true even if the sensible heat is small compared to the latent heat of fusion. Similarly, curvature plays a minor role in the interface motion for the insulated case while its effect is dramatic in the isothermal case.

NOMENCLATURE

a,	radius of outer cylinder;
A,	integration constant;
b,	radius of inner cylinder;
B,	integration constant;
C,	specific heat;
K,	thermal conductivity;
L,	latent heat of fusion;
n,	perturbation order;
r,	radial coordinate;
r_i,	interface location;
t,	long time variable;
$\mathcal{T}$,	short time variable;
T,	temperature;
T_a,	temperature at boundary;
T_f,	fusion temperature;
T_i,	initial temperature of the liquid phase.

Greek symbols

α,	thermal diffusivity, ratio of solid to liquid-phase thermal diffusivity;
β,	ratio of inner to outer cylinder radius;
ε,	perturbation parameter;
η,	dimensionless radial coordinate in the liquid phase;
θ,	dimensionless solid-phase temperature;
ξ,	dimensionless radial coordinate in the solid-phase;
ρ,	density;
σ,	dimensionless interface location;
τ,	dimensionless long time;
ϕ,	dimensionless liquid-phase temperature;
ϕ_i,	dimensionless initial liquid-phase temperature.

Subscripts

0, 1, 2,	zero, first and second-order perturbation;
l,	liquid;
s,	solid.

Superscripts

inner expansion.

1. INTRODUCTION

A LARGE number of technically important problems involve solutions of the equations describing diffusion of heat, mass, or some other scalar quantity subject to boundaries that are neither fixed in space nor known *a priori*. Examples of such problems are found in melting, freezing, casting, welding, ablation and frost formation. Appropriately, much work has been done in treating these free boundary multiphase problems. Boley [1], Bankoff [2], Muehlbauer and Sunderland [3], Rubinstein [4] and Fox [5] cite many references in their comprehensive literature surveys.

The inherent difficulty in the analysis of such problems is the nonlinear nature of the interface boundary condition which precludes superposition and necessitates the use of special solution techniques. Another difficulty arises in finite domain problems where self-similar solutions cannot be constructed. Because of these two mathematical difficulties, only the simplest type of free boundary problem has been considered. Neumann's [6] solution to the problems of freezing in a semi-infinite region, which was presented in the 1860's remains one of few exact solutions available.

Understandably, most models used in the analysis of multiphase free boundary problems are highly simplified, often limited to one-dimensional systems with simple geometries and boundary conditions which lend themselves to analytical treatment and experimental simulation. Two common simplifications are (i) the assumption of quasi-steady behavior and (ii) the restriction to an initial temperature which is equal to the fusion temperature. Unfortunately, while this last limitation simplifies the mathematical complexity, it fails to represent problems of practical interest. Recently a number of investigators have focused on non-similar free boundary problems in which the phase ahead of the advancing interface is at the fusion temperature or surface concentration. Riley, Smith and Poots [7], Tao [8], and Pedroso and Domoto [9] have examined the inward solidification of cylinders and spheres, Jiji [10] and Shih and Tsay [11] the outward growth and decay of a solid phase on a cylindrical surface, Duda and Vrentas [12, 13], the growth and dissolution of a spherical bubble and Cho and Sunderland [14], Shih and Chou [15], and Theofanoas and Lim [16] the solidification of a saturated liquid outside a sphere.

For free boundary problems in finite domains which are initially at the fusion temperature it is possible to introduce a single coordinate transformation in which the moving interface is immobilized by scaling the independent distance coordinate by the instantaneous interface location. This transformation shifts the nonlinearity from the interface boundary conditions to the governing differential equation for the solid phase which in this case remains a partial differential equation since a similarity solution does not exist. Duda and Vrentas [12, 17] applied this procedure for both cylinders and spheres where one of the phases is at the fusion temperature or interface concentration. For the more general case where both phases are of finite extent with the initial temperature not equal to the fusion temperature, a double coordinate transformation is required since a different instantaneous length is necessary for each region. This double transformation technique is developed in Weinbaum and Jiji [18] and applied to the solution of freezing in finite slabs.

This paper presents perturbation solutions for the problem of inward solidification in an annulus which is initially not at the fusion temperature. The outside radius of the annulus is maintained at a sub-freezing temperature while the inside radius is either insulated or maintained at constant temperature above the fusion level.

A double transformation is used to immobilize the interface and the resulting equations are solved by the method of singular perturbation. The perturbation parameter ε used in the solution is defined as $\varepsilon = C_s(T_f - T_a)/L$ where C_s is the specific heat of the solid phase, T_f the fusion temperature, T_a the temperature of the outer boundary of the annulus and L is the latent heat of fusion. Thus ε is the ratio of the sensible heat of the solid phase to the latent heat of fusion. Typical values of C_s/L per °C for representative materials are 0.006 for ice, 0.00163 for iron and 0.005 for lead. Since perturbation solutions are valid for $\varepsilon < 1$, substantially high values of $(T_f - T_a)$ can be tolerated without compromising the accuracy of the solution.

The perturbation solution presented here is motivated by the nature of quasi-steady and quasi-stationary solutions to free boundary problems in finite domains previously obtained by other investigators. In the quasi-steady approximation the unsteady term is omitted in the differential equation and the gradient at the interface is determined by solving the steady state diffusion equation with a stationary interface. Representative solutions are the dissolution of a gas bubble obtained by Bankoff [2] and Rosner [19]. In general, quasi-steady solutions are not capable of satisfying initial conditions. In the quasi-stationary approximation the unsteady term is retained in the diffusion equation and the latter solved assuming that the interface is stationary. Duda and Vrentas [17] have shown that the quasi-stationary approximation is the leading term of a series solution based on regular perturbation procedure for the finite region problems with only one non-uniform phase. In retaining the unsteady term in the diffusion equation, the quasi-stationary approximation is capable of satisfying an initial temperature or concentration profile. However, the solution in general will not be valid for all time if the temperature ahead of the advancing interface is not uniform and is changing due to the interface motion.

The properties of the quasi-steady and quasi-stationary approximations just described suggest that it might be possible to construct a composite series solution in the perturbation parameter ε which is uniformly valid for all time as a matched asymptotic expansion. In such an expansion the quasi-steady and quasi-stationary approximation serve as the lowest order generating functions in a scheme of successive asymptotic approximations. One anticipates that the composite solution has a boundary-layer-like structure, that is to say, a long-time behavior in which the interface motion is a slowly varying function of time and the temperature or concentration profiles are close to an instantaneous quasi-steady state distribution in some asymptotic sense, and a short time behavior in which the interface motion is rapidly varying and the temperature or concentration profiles are also rapidly changing so as to satisfy appropriate initial conditions. The reason for introducing two separate time scales in the analysis is that different mathematical simplifications obtain on the short and long time scales. On the short time scale one is able to simplify the interface boundary condition and satisfy the initial conditions whereas on the long-time scale one is able to neglect the initial conditions and simplify the governing differential equations but satisfy the nonlinear interface conditions. Two separate solutions of different character emerge in the form of infinite series in fractional half powers of ε which when matched term by term in a region of overlapping validity provide a composite solution which is uniformly valid for all time.

2. FORMULATION

The two problems considered are shown in Fig. 1. The liquid in the annular space is initially at temperature T_i which is above the freezing temperature T_f. At time $t = 0$ the outer tube boundary at $r = a$ is suddenly maintained at a sub-freezing temperature T_a. In the first problem the inner tube boundary at $r = b$ is assumed to be perfectly insulated while in the second problem this boundary is maintained at temperature T_i. These two boundary-value problems are distinguished only by the boundary condition at $r = b$ but they exhibit a markedly different behavior as time progresses. When the inner tube is insulated the entire liquid in the annular space will eventually solidify. On the other hand when the inner tube is maintained at a constant temperature, a steady state is reached in which only a portion of the liquid solidifies.

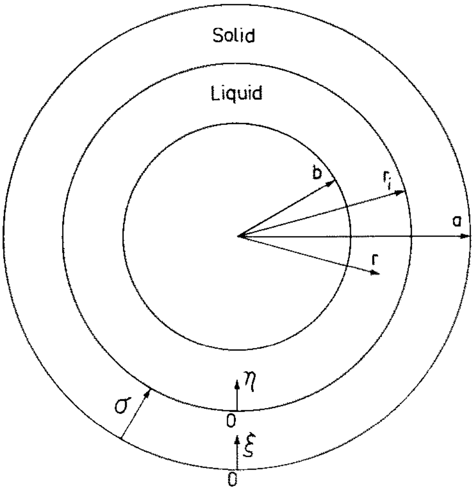

FIG. 1. Configuration and coordinates.

To simplify the problem, the fluid properties in each phase will be assumed constant. Volumetric expansion or contraction due to phase transformation will be neglected. Free convection currents in the liquid phase will be ignored. Based on these simplifications the energy equation in the solid and liquid phases become:

$$\alpha_s\left(\frac{\partial^2 T_s}{\partial r^2}+\frac{1}{r}\frac{\partial T_s}{\partial r}\right)=\frac{\partial T_s}{\partial t},\quad r_i(t)<r<a,$$

and

$$\alpha_l\left(\frac{\partial^2 T_l}{\partial r^2}+\frac{1}{r}\frac{\partial T_l}{\partial r}\right)=\frac{\partial T_l}{\partial t},\quad b<r<r_i(t),$$

where the subscripts s and l denote solid and liquid phase, respectively, and α is the thermal diffusivity. The boundary conditions for the two problems considered are

$$T_s(a,t) = T_a,$$

$$T_s(r_i,t) = T_l(r_i,t) = T_f,$$

$$\frac{\partial T_l(b,t)}{\partial r} = 0 \quad \text{or} \quad T_l(b,t) = T_i.$$

The initial conditions on the liquid temperature T_l and the interface location $r_i(t)$ are

$$T_l(r,0) = T_i,$$

$$r_i(0) = a.$$

The energy balance at the interface gives

$$K_s\frac{\partial T_s(r_i,t)}{\partial r}-K_l\frac{\partial T_l(r_i,t)}{\partial r}=\rho L\frac{\mathrm{d}r_i}{\mathrm{d}t},$$

where ρ is the mass density.

To non-dimensionalize the governing equations and immobilize the boundaries, the following dimensionless quantities are defined:

$$\left.\begin{aligned}
\eta &= \frac{r_i(t)-r}{r_i(t)-b}\\
\xi &= \frac{a-r}{a-r_i(t)}\\
\tau &= \varepsilon\alpha_s t/a^2\\
\theta &= (T_s-T_f)/(T_f-T_a)\\
\phi &= K_l(T_l-T_f)/(K_s(T_f-T_a))\\
\sigma &= (a-r_i)/a\\
\alpha &= \alpha_s/\alpha_l\\
\beta &= b/a\\
\varepsilon &= C_s(T_f-T_a)/L.
\end{aligned}\right\}\qquad (1)$$

Two dimensionless distance coordinates η and ξ are required to immobilize the interface location and the inner tube boundary. This is an important distinction between the present analysis and the theory in [12], [13] and [17] where the region ahead of the advancing interface is at the fusion temperature. The reference time a^2/α_s is the characteristic diffusion time for the solid phase, whereas the characteristic time used in the definition of the dimensionless time τ is $a^2/\varepsilon\alpha_s$. This latter time is long compared to the solid diffusion time if $\varepsilon < 1$ and represents the characteristic time for the interface motion to occur. θ, ϕ and σ are the dimensionless temperatures in the solid and liquid phases and dimensionless interface location, respectively.

Using the dimensionless quantities defined in (1), the governing equations and boundary conditions become:

$$\frac{\partial^2\theta}{\partial\xi^2}-\frac{\sigma}{1-\sigma\xi}\frac{\partial\theta}{\partial\xi}=\varepsilon\left\{\sigma^2\frac{\partial\theta}{\partial\tau}-\sigma\xi\frac{\mathrm{d}\sigma}{\mathrm{d}\tau}\frac{\partial\theta}{\partial\xi}\right\},\quad 0<\xi<1,\qquad (2)$$

$$\frac{\partial^2\phi}{\partial\eta^2}-\frac{(1-\sigma-\beta)}{[(1-\sigma)-(1-\sigma-\beta)\eta]}\frac{\partial\phi}{\partial\eta}$$

$$=\alpha\varepsilon\left\{[(1-\sigma-\beta)]^2\frac{\partial\phi}{\partial\tau}-(1-\sigma-\beta)(1-\eta)\frac{\mathrm{d}\sigma}{\mathrm{d}\tau}\frac{\partial\phi}{\partial\eta}\right\},\quad 0<\eta<1,\qquad (3)$$

$$\theta(0,\tau) = -1, \qquad (4)$$

$$\theta(1,\tau) = 0, \qquad (5)$$

$$\phi(0,\tau) = 0, \qquad (6)$$

$$\frac{\partial\phi(1,\tau)}{\partial\eta} = 0, \quad \text{(insulated case)} \qquad (7a)$$

or

$$\phi(1,\tau) = \phi_i, \quad \text{(isothermal case)} \tag{7b}$$

$$\phi(\eta,0) = \phi_i, \tag{8}$$

$$\sigma(0) = 0, \tag{9}$$

$$(1-\sigma-\beta)\frac{\partial\theta(1,\tau)}{\partial\xi} - \sigma\frac{\partial\phi(0,\tau)}{\partial\eta} = \sigma(1-\sigma-\beta)\frac{\mathrm{d}\sigma}{\mathrm{d}\tau}. \tag{10}$$

Examination of equations (2)–(10) shows that the free boundary problem has been transformed to a stationary problem and the nonlinearity due to the unknown interface motion $r_i(t)$ is shifted from the interface boundary conditions to the differential equations (2) and (3). Furthermore, the problem is governed by four parameters α, β, ε and ϕ_i. The thermal conductivity ratio K_l/K_s is eliminated as a parameter by including it in the definition of the liquid temperature ϕ. The thermal diffusivity ratio $\alpha = \alpha_s/\alpha_l$ is approximately equal to unity for most materials. The geometry parameter $\beta = b/a$ is a measure of the curvature of the annular space. The special case of $\beta = 0$ represents solidification in a tube. For this case, only the insulated inner boundary problem is physically possible.

3. PERTURBATION SOLUTIONS

Outer expansion

We seek approximate solutions to equations (2) and (3), subject to boundary and initial conditions (4)–(10), in the form of an asymptotic expansion valid for $\varepsilon < 1$. A cursory examination of equations (2) and (3) shows that a regular perturbation expansion of the form

$$\theta(\xi,\tau;\varepsilon) \sim \sum_{n=0}^{N} \varepsilon^{n/2}\theta_n(\xi,\tau), \tag{11}$$

$$\phi(\eta,\tau;\varepsilon) \sim \sum_{n=0}^{N} \varepsilon^{n/2}\phi_n(\eta,\tau), \tag{12}$$

and

$$\sigma(\tau;\varepsilon) \sim \sum_{n=0}^{N} \varepsilon^{n/2}\sigma_n(\tau), \tag{13}$$

is singular as $\tau \to 0$. The time derivative terms containing $\partial\theta_0/\partial\tau$ and $\partial\phi_0/\partial\tau$ in the lowest order differential equation for (2) and (3) are lost and initial condition (8) cannot be satisfied. The nonuniformity of the expansions (11)–(13) in the domain near $\tau = 0$ will be corrected by constructing inner expansions for θ, ϕ and σ valid on a time scale where $\tau \ll 1$. These inner expansions will satisfy the initial conditions (8) and (9) and match in a region of overlapping validity with the series solutions (11)–(13) which is the large time or outer expansion.

The zero, first and second order outer expansions are constructed by substituting (11)–(13) into (2)–(10) and equating terms of identical powers of ε. The governing equations and boundary conditions will thus be formulated and solutions for the two cases under consideration will be presented.

(i) *Zero-order expansion.* The zero-order solid and liquid phase equations are

$$(1-\sigma_0\xi)\frac{\partial^2\theta_0}{\partial\xi^2} - \sigma_0\frac{\partial\theta_0}{\partial\xi} = 0, \quad 0 < \xi < 1, \tag{14}$$

and

$$[(1-\eta)(1-\sigma_0)+\beta\eta]\frac{\partial^2\phi_0}{\partial\eta^2} - (1-\sigma_0-\beta)\frac{\partial\phi_0}{\partial\eta} = 0, \quad 0 < \eta < 1. \tag{15}$$

The boundary conditions are

$$\theta_0(0,\tau) = -1, \tag{16}$$

$$\theta_0(1,\tau) = 0, \tag{17}$$

$$\phi_0(0,\tau) = 0, \tag{18}$$

$$\frac{\partial\phi_0(1,\tau)}{\partial\eta} = 0, \tag{19a}$$

or

$$\phi_0(1,\tau) = \phi_i. \tag{19b}$$

The interface condition is

$$(1-\sigma_0-\beta)\frac{\partial\theta_0(1,\tau)}{\partial\xi} - \sigma_0\frac{\partial\phi_0(0,\tau)}{\partial\eta} = \sigma_0(1-\sigma_0-\beta)\frac{\mathrm{d}\sigma_0}{\mathrm{d}\tau}. \tag{20}$$

The solid phase temperature solution θ_0 which is valid for both the insulated and the isothermal boundary condition at $\eta = 1$ is

$$\theta_0 = \frac{\ln(1-\sigma_0\xi)}{\ln(1-\sigma_0)} - 1. \tag{21}$$

For the insulated case, the liquid phase solution ϕ_0 is

$$\phi_0 = 0. \tag{22a}$$

The corresponding solution for the isothermal case is

$$\phi_0 = \frac{\phi_i}{\ln\dfrac{(1-\sigma_0)}{\beta}}\ln\frac{(1-\sigma_0)}{(1-\eta)(1-\sigma_0)+\beta\eta}. \tag{22b}$$

The interface solution σ_0 is obtained by substituting (21) and (22) into (20). For the insulated case σ_0 is given by

$$(1-\sigma_0)^2[\ln(1-\sigma_0)-\tfrac{1}{2}] = 2(A_0+\tau), \tag{23a}$$

where A_0 is the integration constant which will be determined later by matching with the inner solution.

For the isothermal case the differential equation for σ_0 is

$$\mathrm{d}\tau = \frac{(1-\sigma_0)\,\mathrm{d}\sigma_0}{\phi_i/\ln\left(\dfrac{\beta}{1-\sigma_0}\right) - 1/\ln(1-\sigma_0)}. \tag{24}$$

Integration of (24) gives

$$\begin{aligned} B_0 - (1+\phi_i)\beta^{-2/(1+\phi_i)}\tau &= \phi_i\left(\frac{\ln\beta}{1+\phi_i}\right)^2\left\{\ln\,\ln\frac{\beta^{1/(1+\phi_i)}}{1-\sigma_0}\right. \\ &\quad + \left.\sum_{j=1}^{\infty}\frac{[2\ln(1-\sigma_0)\beta^{-1/(1+\phi_i)}]^j}{j\cdot j!}\right\} \\ &\quad - \tfrac{1}{2}[\ln(1-\sigma_0)\beta^{-1/(1+\phi_i)} - \tfrac{1}{2}] \\ &\quad \times (1-\sigma_0)^2\beta^{-2/(1+\phi_i)}, \end{aligned} \tag{23b}$$

where B_0 is the constant of integration.

(ii) *First-order expansion.* The governing equations and boundary conditions for the first order problem are

$$(1-\sigma_0\xi)\frac{\partial^2\theta_1}{\partial\xi^2}-\sigma_1\xi\frac{\partial^2\theta_0}{\partial\xi^2}-\sigma_0\frac{\partial\theta_1}{\partial\xi}-\sigma_1\frac{\partial\theta_0}{\partial\xi}=0, \quad (25)$$

$$(1-\eta)\Big|\,(1-\sigma_0)\frac{\partial^2\phi_1}{\partial\eta^2}-\sigma_1\frac{\partial^2\phi_0}{\partial\eta^2}+\beta\eta\frac{\partial^2\phi_1}{\partial\eta^2}$$

$$-(1-\sigma_0-\beta)\frac{\partial\phi_1}{\partial\eta}+\sigma_1\frac{\partial\phi_0}{\partial\eta}=0, \quad (26)$$

$$\theta_1(0,\tau)=\theta_1(1,\tau)=\phi_1(0,\tau)=0, \quad (27)$$

$$\frac{\partial\phi_1(1,\tau)}{\partial\eta}=0, \quad (28a)$$

or

$$\phi_1(1,\tau)=0, \quad (28b)$$

$$(1-\sigma_0-\beta)\frac{\partial\theta_1(1,\tau)}{\partial\xi}-\sigma_1\frac{\partial\theta_0(1,\tau)}{\partial\xi}$$

$$-\sigma_0\frac{\partial\phi_1(0,\tau)}{\partial\eta}-\sigma_1\frac{\partial\phi_0(0,\tau)}{\partial\eta}$$

$$=\sigma_0(1-\sigma_0-\beta)\frac{\mathrm{d}\sigma_1}{\mathrm{d}\tau}-\sigma_0\sigma_1\frac{\mathrm{d}\sigma_0}{\mathrm{d}\tau}$$

$$+\sigma_1(1-\sigma_0-\beta)\frac{\mathrm{d}\sigma_0}{\mathrm{d}\tau}. \quad (29)$$

The temperature solution for the solid phase is

$$\theta_1=\frac{\sigma_1}{\ln(1-\sigma_0)}\Big|\,\frac{\ln(1-\sigma_0\xi)}{(1-\sigma_0)\ln(1-\sigma_0)}-\frac{\xi}{1-\sigma_0\xi}\,\Big|. \quad (30)$$

Equation (30) satisfies both the insulated and isothermal conditions at $\eta = 1$. The liquid phase solution for the insulated case is

$$\phi_1=0. \quad (31a)$$

The corresponding solution for the isothermal case is

$$\phi_1=\frac{\phi_1\sigma_1}{(1-\sigma_0)\ln\dfrac{(1-\sigma_0)}{\beta}}$$

$$\times\quad\frac{1}{\ln\dfrac{(1-\sigma_0)}{\beta}}\ln\frac{(1-\sigma_0)}{(1-\eta)(1-\sigma_0)+\beta\eta}$$

$$-\frac{\beta\eta}{(1-\eta)(1-\sigma_0)+\beta\eta}\ . \quad (31b)$$

The interface solution σ_1 for the insulated case is

$$\sigma_1=\frac{A_1}{(1-\sigma_0)\ln(1-\sigma_0)}. \quad (32a)$$

For the isothermal case the differential equation (29) for σ_1 cannot be integrated to give an exact solution. However, since $\sigma_0 < 1$ for small values of the large time variable and $0 < \beta < 1$, an approximate solution for σ_1 for $\tau \ll 1$ can be obtained for $|(1+\phi_i)\sigma_0/\ln\beta| \ll 1$. This solution is given by

$$\sigma_1=B_1/\sigma_0, \quad (32b)$$

where B_1 is the integration constant.

(iii) *Second-order expansion.* The governing equations for the second order problem are considerably simplified if we take advantage of the matching presented later in this section which shows that $A_1 = B_1 = 0$ and consequently $\theta_1 = \phi_1 = \sigma_1 = 0$. Based on this observation the governing equations and boundary conditions become

$$(1-\sigma_0\xi)\frac{\partial^2\theta_2}{\partial\xi^2}-\sigma_2\xi\frac{\partial^2\theta_0}{\partial\xi^2}-\sigma_0\frac{\partial\theta_2}{\partial\xi}-\sigma_2\frac{\partial\theta_0}{\partial\xi}$$

$$=\sigma_0^2(1-\sigma_0\xi)\frac{\partial\theta_0}{\partial\tau}+\sigma_0\xi(\sigma_0\xi-1)\frac{\mathrm{d}\sigma_0}{\mathrm{d}\tau}\frac{\partial\theta_0}{\partial\xi}, \quad (33)$$

$$[(1-\eta)(1-\sigma_0)+\beta\eta]\frac{\partial^2\phi_2}{\partial\eta^2}-(1-\beta-\sigma_0)\frac{\partial\phi_2}{\partial\eta}$$

$$=0, \quad (34)$$

$$\theta_2(0,\tau)=\theta_2(1,\tau)=\phi_2(0,\tau)=0, \quad (35)$$

$$\partial\phi_2(1,\tau)/\partial\eta=0 \quad (36a)$$

$$(1-\sigma_0-\beta)\frac{\partial\theta_2(1,\tau)}{\partial\xi}-\sigma_2\frac{\partial\theta_0(1,\tau)}{\partial\xi}-\sigma_0\frac{\partial\phi_2(0,\tau)}{\partial\eta}$$

$$=\sigma_2(1-2\sigma_0-\beta)\frac{\mathrm{d}\sigma_0}{\mathrm{d}\tau}+\sigma_0(1-\sigma_0-\beta)\frac{\mathrm{d}\sigma_2}{\mathrm{d}\tau}. \quad (37)$$

The temperature solution θ_2 in the solid phase is

$$\theta_2=\frac{\sigma_2}{\ln(1-\sigma_0)}\Big|\,\frac{1}{\sigma_0}-\frac{1}{\sigma_0(1-\sigma_0\xi)}+\frac{\ln(1-\sigma_0\xi)}{(1-\sigma_0)\ln(1-\sigma_0)}\,\Big|$$

$$-\frac{1}{4(1-\sigma_0)^2[\ln(1-\sigma_0)]^3}\,\Big|\,(1-\sigma_0\xi)^2\ln(1-\sigma_0\xi)$$

$$+2\sigma_0\xi-\sigma_0^2\xi^2+\frac{\sigma_0^2\ln(1-\sigma_0\xi)}{\ln(1-\sigma_0)}-\frac{2\sigma_0\ln(1-\sigma_0\xi)}{\ln(1-\sigma_0)}$$

$$-(1-\sigma_0)^2\ln(1-\sigma_0\xi)\,\Big|. \quad (38)$$

Equation (38) is valid for both the insulated and isothermal boundary condition at $\eta = 1$. The solutions to ϕ_2 and σ_2 for the insulated case are

$$\phi_2=0, \quad (39a)$$

and

$$\sigma_2=\frac{\sigma_0(1-\sigma_0/2)}{2(1-\sigma_0)\ln(1-\sigma_0)}\Big|\,\frac{1}{\ln(1-\sigma_0)}-1\,\Big|$$

$$-\frac{A_2}{(1-\sigma_0)\mathit{ln}(1-\sigma_0)}. \quad (40a)$$

The second order solutions for ϕ_2 and σ_2 for the isothermal case have not been obtained because the corresponding differential equations cannot be integrated analytically.

The outer expansion solutions are thus equivalent to a quasi-steady approximation. They satisfy the nonlinear interface equation (10) to order $\varepsilon^{1/2}$ for the isothermal case and to order ε for the insulated case. However, the initial conditions (8) and (9) are not satisfied. To construct a composite solution valid for all time we shall introduce a stretched time coordinate $\mathcal{T} = \tau/\varepsilon$ which is of order unity when τ is small and of order ε. It is on this stretched inner or small time scale that the transient behavior occurs through which the temperature distribution in the liquid phase adjusts from its initial distribution (8) to the nearly quasi-steady state distribution given by (22) and (31). For uniform validity the behavior of the outer solutions for θ and σ must match term by term as τ approaches zero with the behavior of the inner solution as $\mathcal{T}$ approaches infinity.

Inner expansion

In accord with the foregoing remarks we introduce the following inner variables with magnified time scale:

$$\hat{\theta}(\xi, \mathcal{T}) = \theta(\xi, \tau),$$
$$\hat{\phi}(\eta, \mathcal{T}) = \phi(\eta, \tau),$$
$$\hat{\sigma}(\mathcal{T}) = \sigma(\tau),$$
$$\mathcal{T} = \tau/\varepsilon.$$

From the definition of the dimensionless long time variable τ it is evident that the reference time for the scaling of the dimensionless short-time variable $\mathcal{T}$ is a^2/α_s, the characteristic diffusion time for the solid phase. Substituting the inner variables into the governing equations and boundary conditions (2)–(10), we obtain

$$\frac{\partial^2\hat{\theta}}{\partial\xi^2} - \frac{\hat{\sigma}}{1-\hat{\sigma}\xi}\frac{\partial\hat{\theta}}{\partial\xi} = \hat{\sigma}^2\frac{\partial\hat{\theta}}{\partial\mathcal{T}} - \hat{\sigma}\xi\frac{\mathrm{d}\hat{\sigma}}{\mathrm{d}\mathcal{T}}\frac{\partial\theta}{\partial\xi}, \tag{41}$$

$$\frac{\partial^2\hat{\phi}}{\partial\eta^2} - \frac{(1-\hat{\sigma}-\beta)}{[(1-\hat{\sigma})-(1-\hat{\sigma}-\beta)\eta]}\frac{\partial\hat{\phi}}{\partial\eta}$$
$$= \alpha\left\{(1-\hat{\sigma}-\beta)^2\frac{\partial\hat{\phi}}{\partial\mathcal{T}} - (1-\hat{\sigma}-\beta)(1-\eta)\frac{\mathrm{d}\hat{\sigma}}{\mathrm{d}\mathcal{T}}\frac{\partial\hat{\phi}}{\partial\eta}\right\} \tag{42}$$

$$\hat{\theta}(0, \mathcal{T}) = -1, \tag{43}$$

$$\hat{\theta}(1, \mathcal{T}) = 0, \tag{44}$$

$$\hat{\phi}(0, \mathcal{T}) = 0, \tag{45}$$

$$\frac{\partial\hat{\phi}(1, \mathcal{T})}{\partial\eta} = 0, \tag{46a}$$

or

$$\hat{\phi}(1, \mathcal{T}) = \phi_i, \tag{46b}$$

$$\hat{\phi}(\eta, 0) = \phi_i, \tag{47}$$

$$\hat{\sigma}(0) = 0, \tag{48}$$

$$\varepsilon\left[(1-\hat{\sigma}-\beta)\frac{\partial\hat{\theta}(1, \mathcal{T})}{\partial\xi} - \hat{\sigma}\frac{\partial\hat{\phi}(0, \mathcal{T})}{\partial\eta}\right]$$
$$= \hat{\sigma}(1-\hat{\sigma}-\beta)\frac{\mathrm{d}\hat{\sigma}}{\mathrm{d}\mathcal{T}}. \tag{49}$$

To solve the system of equations (41)–(49), we assume inner expansions for $\hat{\theta}$, $\hat{\phi}$ and $\hat{\sigma}$ of the form

$$\hat{\theta}(\xi, \mathcal{T}; \varepsilon) \sim \sum_{n=0}^{N} \varepsilon^{n/2}\hat{\theta}_n(\xi, \mathcal{T}), \tag{50}$$

$$\hat{\phi}(\eta, \mathcal{T}; \varepsilon) \sim \sum_{n=0}^{N} \varepsilon^{n/2}\hat{\phi}_n(\eta, \mathcal{T}), \tag{51}$$

and

$$\hat{\sigma}(\mathcal{T}; \varepsilon) \sim \sum_{n=0}^{N} \varepsilon^{n/2}\hat{\sigma}_n(\mathcal{T}). \tag{52}$$

We proceed by substituting (50)–(52) into (41)–(49) and equating coefficients of like powers of ε. The resulting equations simplify greatly if one first observes that the LHS of (49) is order ε smaller than the RHS and consequently the solution for the zero-order interface location $\hat{\sigma}_0$ which satisfies the initial condition (48) is $\hat{\sigma}_0 = 0$.

(i) *Zero-order expansion.* With $\hat{\sigma}_0 = 0$, the zero-order governing equations and boundary and initial conditions become:

$$\frac{\partial^2\hat{\theta}_0}{\partial\xi^2} = 0, \tag{53}$$

$$[1-(1-\beta)\eta]\frac{\partial^2\hat{\phi}_0}{\partial\eta^2} - (1-\beta)\frac{\partial\hat{\phi}_0}{\partial\eta}$$
$$= \alpha[(1-\beta)^2(1-\eta+\beta\eta)]\frac{\partial\hat{\phi}_0}{\partial\mathcal{T}}, \tag{54}$$

$$\hat{\theta}_0(0, \mathcal{T}) = -1, \tag{55}$$

$$\hat{\theta}_0(1, \mathcal{T}) = 0, \tag{56}$$

$$\hat{\phi}_0(0, \mathcal{T}) = 0, \tag{57}$$

$$\frac{\partial\hat{\phi}_0(1, \mathcal{T})}{\partial\eta} = 0, \tag{58}$$

or

$$\hat{\phi}_0(1, \mathcal{T}) = \phi_i, \tag{59}$$

$$\hat{\phi}_0(\eta, 0) = \phi_i, \tag{60}$$

$$\hat{\sigma}_0(0) = 0, \tag{61}$$

$$\hat{\sigma}_0(1-\hat{\sigma}_0-\beta)\frac{\mathrm{d}\hat{\sigma}_0}{\mathrm{d}\mathcal{T}} = 0. \tag{62}$$

The solutions to $\hat{\theta}_0$ and $\hat{\sigma}_0$ are

$$\hat{\theta}_0 = \xi - 1, \tag{63}$$

and

$$\hat{\sigma}_0 = 0. \tag{64}$$

Both (63) and (64) are valid for the insulated and the isothermal cases. However, the liquid temperature $\hat{\phi}_0$ depends on the boundary condition at $\eta = 1$. For the insulated case we obtain

$$\hat{\phi}_0 = \sum_{K=0}^{\infty} a_K \left[J_0\left[\frac{\lambda_K}{1-\beta}(1-\eta+\beta\eta)\right] - \frac{J_1\left(\frac{\beta\lambda_K}{1-\beta}\right)}{Y_1\left(\frac{\beta\lambda_K}{1-\beta}\right)} Y_0\left[\frac{\lambda_K}{1-\beta}(1-\eta+\beta\eta)\right]\right] \mathrm{e}^{-[\lambda_K^2/\alpha(1-\beta)^2]\mathcal{T}}, \tag{65a}$$

where λ_K is given by

$$J_0\left(\frac{\lambda_K}{1-\beta}\right) = Y_0\left(\frac{\lambda_K}{1-\beta}\right)\frac{J_1\left(\frac{\beta\lambda_K}{1-\beta}\right)}{Y_1\left(\frac{\beta\lambda_K}{1-\beta}\right)}, \tag{66a}$$

and

$$a_K = \frac{(1-\beta)\phi_i}{\lambda_K}\left[\frac{J_1\left(\frac{\beta\lambda_K}{1-\beta}\right)}{Y_1\left(\frac{\beta\lambda_K}{1-\beta}\right)} Y_1\left(\frac{\lambda_K}{1-\beta}\right) - J_1\left(\frac{\lambda_K}{1-\beta}\right)\right] \left\{\frac{\beta^2}{2} J_0^2\left(\frac{\beta\lambda_K}{1-\beta}\right)\right.$$

$$-\tfrac{1}{2}J_1^2\left(\frac{\lambda_K}{1-\beta}\right) + \frac{J_1^2\left(\frac{\beta\lambda_K}{1-\beta}\right)}{Y_1^2\left(\frac{\beta\lambda_K}{1-\beta}\right)}\left|\frac{\beta^2}{2} Y_0^2\left(\frac{\beta\lambda_K}{1-\beta}\right) - \tfrac{1}{2} Y_1^2\left(\frac{\lambda_K}{1-\beta}\right)\right]$$

$$\left. - \frac{J_1\left(\frac{\beta\lambda_K}{1-\beta}\right)}{Y_1\left(\frac{\beta\lambda_K}{1-\beta}\right)}\left| \beta^2 J_0\left(\frac{\beta\lambda_K}{1-\beta}\right) Y_0\left(\frac{\beta\lambda_K}{1-\beta}\right) - J_1\left(\frac{\lambda_K}{1-\beta}\right) Y_1\left(\frac{\lambda_K}{1-\beta}\right)\right| \right\} \tag{67a}$$

The corresponding solution for the isothermal case is

$$\hat{\phi}_0 = \frac{\phi_i \ln[1-(1-\beta)\eta]}{\ln\beta} + \sum_{K=0}^{\infty} b_K \left\{ J_0\left|\frac{\gamma_K}{1-\beta}(1-\eta+\beta\eta)\right|\right.$$

$$\left. - \frac{J_0\left(\frac{\beta\gamma_K}{1-\beta}\right)}{Y_0\left(\frac{\beta\gamma_K}{1-\beta}\right)} Y_0\left|\frac{\gamma_K}{1-\beta}(1-\eta+\beta\eta)\right| \right\} \mathrm{e}^{-[\gamma_K^2/\alpha(1-\beta)^2]\mathscr{T}}, \tag{65b}$$

where γ_K is given by

$$J_0\left(\frac{\gamma_K}{1-\beta}\right) = \frac{J_0\left(\frac{\beta\gamma_K}{1-\beta}\right)}{Y_0\left(\frac{\beta\gamma_K}{1-\beta}\right)} Y_0\left(\frac{\gamma_K}{1-\beta}\right), \tag{66b}$$

and

$$b_K = \frac{\phi_i(1-\beta)}{\gamma_K}\left[\frac{J_0\left(\frac{\beta\gamma_K}{1-\beta}\right)}{Y_0\left(\frac{\beta\gamma_K}{1-\beta}\right)} Y_1\left(\frac{\gamma_K}{1-\beta}\right) - J_1\left(\frac{\gamma_K}{1-\beta}\right)\right] \left\{\frac{\beta^2}{2} J_1^2\left(\frac{\beta\gamma_K}{1-\beta}\right)\right.$$

$$-\tfrac{1}{2}J_1^2\left(\frac{\gamma_K}{1-\beta}\right) + \frac{J_0^2\left(\frac{\beta\gamma_K}{1-\beta}\right)}{Y_0^2\left(\frac{\beta\gamma_K}{1-\beta}\right)}\left|\frac{\beta^2}{2} Y_1^2\left(\frac{\beta\gamma_K}{1-\beta}\right) - \tfrac{1}{2} Y_1^2\left(\frac{\gamma_K}{1-\beta}\right)\right|$$

$$\left. - \frac{J_0\left(\frac{\beta\gamma_K}{1-\beta}\right)}{Y_0\left(\frac{\beta\gamma_K}{1-\beta}\right)}\left| \beta^2 J_1\left(\frac{\beta\gamma_K}{1-\beta}\right) Y_1\left(\frac{\beta\gamma_K}{1-\beta}\right) - J_1\left(\frac{\gamma_K}{1-\beta}\right) Y_1\left(\frac{\gamma_K}{1-\beta}\right)\right| \right\}. \tag{67b}$$

(ii) *First-order expansion.* The determination of the first order liquid-phase temperature solution $\hat{\phi}_1$ is too complicated and difficult to obtain. Fortunately, the interface motion $\hat{\sigma}_1$ can still be obtained without knowing $\hat{\phi}_1$. We therefore proceed with the governing equations and boundary and initial conditions for $\hat{\theta}_1$ and $\hat{\sigma}_1$:

$$\frac{\partial^2\hat{\theta}_1}{\partial\xi^2} - \hat{\sigma}_1\xi\frac{\partial^2\hat{\theta}_0}{\partial\xi^2} - \hat{\sigma}_1\frac{\partial\hat{\theta}_0}{\partial\xi} = 0, \tag{68}$$

$$\hat{\sigma}_0(1-\hat{\sigma}_0-\beta)\frac{\mathrm{d}\hat{\sigma}_1}{\mathrm{d}\mathscr{T}} + \hat{\sigma}_1(1-\beta-2\hat{\sigma}_0)\frac{\mathrm{d}\hat{\sigma}_0}{\mathrm{d}\mathscr{T}} = 0, \tag{69}$$

$$\hat{\theta}_1(0,\mathscr{T}) = \hat{\theta}_1(1,\mathscr{T}) = \hat{\sigma}_1(0) = 0. \tag{70}$$

We note that the interface equation (69) does not yield any information on $\hat{\sigma}_1$ since $\hat{\sigma}_0 = 0$ and (69) reduces to a trivial equation. To determine $\hat{\sigma}_1$ we return to equation (49) and construct the next higher order equation corresponding to terms containing ε. The resulting equation is

$$\frac{\partial\hat{\theta}_0(1,\mathscr{T})}{\partial\xi} = \hat{\sigma}_1\frac{\mathrm{d}\hat{\sigma}_1}{\mathrm{d}\mathscr{T}}. \tag{71}$$

The solution to (71) subject to initial condition (70) gives

$$\hat{\sigma}_1 = (2\mathscr{T})^{1/2}. \tag{72}$$

The solid-phase temperature solution $\hat{\theta}_1$ is obtained from the solution of (68):

$$\hat{\theta}_1 = \tfrac{1}{2}(2\mathscr{T})^{1/2}\xi(\xi-1). \tag{73}$$

It should be noted that both (72) and (73) are valid for the insulated and the isothermal boundary conditions.

(iii) *Second-order expansion.* As in the first-order solution, the interface location $\hat{\sigma}_2$ can be obtained without knowing the first-order liquid-phase solution $\hat{\phi}_1$. The differential equations, boundary and initial conditions for $\hat{\theta}_2$ and $\hat{\sigma}_2$ are:

$$\frac{\partial^2\hat{\theta}_2}{\partial\xi^2} - \hat{\sigma}_1\xi\frac{\partial^2\hat{\theta}_1}{\partial\xi^2} - \hat{\sigma}_2\xi\frac{\partial^2\hat{\theta}_0}{\partial\xi^2} - \hat{\sigma}_1\frac{\partial\hat{\theta}_1}{\partial\xi} - \hat{\sigma}_2\frac{\partial\hat{\theta}_0}{\partial\xi} = -\xi\hat{\sigma}_1\frac{\mathrm{d}\hat{\sigma}_1}{\mathrm{d}\mathscr{T}}\frac{\partial\hat{\theta}_0}{\partial\xi}, \tag{74}$$

$$(1-\beta)\frac{\partial\hat{\theta}_1(1,\mathscr{T})}{\partial\xi} - \hat{\sigma}_1\frac{\partial\hat{\theta}_0(1,\mathscr{T})}{\partial\xi} - \hat{\sigma}_1\frac{\partial\hat{\phi}_0(0,\mathscr{T})}{\partial\eta} = \hat{\sigma}_1(1-\beta)\frac{\mathrm{d}\hat{\sigma}_2}{\mathrm{d}\mathscr{T}} + \hat{\sigma}_2(1-\beta)\frac{\mathrm{d}\hat{\sigma}_1}{\mathrm{d}\mathscr{T}} - \hat{\sigma}_1^2\frac{\mathrm{d}\hat{\sigma}_1}{\mathrm{d}\mathscr{T}}, \tag{75}$$

and

$$\hat{\theta}_2(0,\mathscr{T}) = \hat{\theta}_2(1,\mathscr{T}) = \hat{\sigma}_2(0) = 0. \tag{76}$$

The solid phase temperature solution $\hat{\theta}_2$ which is valid for both the insulated and isothermal cases is

$$\hat{\theta}_2 = \tfrac{1}{6}(4\mathscr{T}-1)\xi^3 + (\hat{\sigma}_2-\mathscr{T})\frac{\xi^2}{2} + \tfrac{1}{6}(1-\mathscr{T})\xi - \frac{\hat{\sigma}_2}{2}\xi. \tag{77}$$

The interface location $\hat{\sigma}_2$ for the insulated case is given by

$$\hat{\sigma}_2 = \frac{\mathscr{T}}{3} + \sum_{K=0}^{\infty} a_K C_K \left| -\mathrm{e}^{-[\lambda_K^2/\alpha(1-\beta)^2]\mathscr{T}} + \frac{(1-\beta)}{2\lambda_K}(\pi\alpha)^{1/2}\frac{1}{(\mathscr{T})^{1/2}}\operatorname{erf}\frac{\lambda_K}{(1-\beta)(\alpha)^{1/2}}(\mathscr{T})^{1/2}\right|, \tag{78a}$$

where

$$C_K = \frac{\alpha(1-\beta)}{\lambda_K}\left[\frac{J_1\left(\dfrac{\beta\lambda_K}{1-\beta}\right)}{Y_1\left(\dfrac{\beta\lambda_K}{1-\beta}\right)}Y_1\left(\frac{\lambda_K}{1-\beta}\right) - J_1\left(\frac{\lambda_K}{1-\beta}\right)\right]. \tag{79a}$$

For the isothermal case the solution is given by

$$\hat{\sigma}_2 = \frac{\mathscr{T}}{3}\left(1+\frac{2\phi_i}{\ln\beta}\right) + \sum_{K=0}^{\infty} b_K D_K \left| -\mathrm{e}^{-[\gamma_K^2/\alpha(1-\beta)^2]\mathscr{T}} + \frac{(1-\beta)(\pi\alpha)^{1/2}}{2\gamma_K}\frac{1}{(\mathscr{T})^{1/2}}\operatorname{erf}\frac{\gamma_K(\mathscr{T})^{1/2}}{(1-\beta)(\alpha)^{1/2}}\right| \tag{78b}$$

where

$$D_K = \frac{\alpha(1-\beta)}{\gamma_K}\left\{\frac{J_0\left(\dfrac{\beta\gamma_K}{1-\beta}\right)}{Y_0\left(\dfrac{\beta\gamma_K}{1-\beta}\right)}Y_1\left(\frac{\gamma_K}{1-\beta}\right) - J_1\left(\frac{\gamma_K}{1-\beta}\right)\right\}. \tag{79b}$$

Matching

To determine the integration constants A_0, A_1, A_2, B_0, B_1, in the outer solutions, the inner and outer expansions for the interface motion σ are matched as $\tau \to 0$ and $\mathscr{T} \to \infty$. The outer and inner expansions for σ are given by

$$\sigma = \sigma_0 + \varepsilon^{1/2}\sigma_1 + \varepsilon\sigma_2 + 0(\varepsilon^{3/2}), \tag{80}$$

and

$$\hat{\sigma} = \hat{\sigma}_0 + \varepsilon^{1/2}\hat{\sigma}_1 + \varepsilon\hat{\sigma}_2 + 0(\varepsilon^{3/2}). \tag{81}$$

We consider first the insulated boundary case. For the outer expansion (80), σ_0, σ_1, and σ_2 are given by (23a), (32a) and (40a), respectively. The inner solutions $\hat{\sigma}_0$, $\hat{\sigma}_1$, and $\hat{\sigma}_2$ are given by (64), (72) and (78a), respectively. The zero-order solution as given by (23a) is not suitable for matching. We first rewrite (23a) using binomial expansions

$$-\sigma_0^2\left(-1+\frac{\sigma_0}{3}+\frac{1}{12}\sigma_0^2\right) = 2\tau + 2A_0 + \tfrac{1}{2}. \tag{82}$$

To facilitate matching we assume that σ_0 can be expressed as

$$\sigma_0 = a_0 + a_1(\tau)^{1/2} + a_2\tau + a_3\tau^{3/2}. \tag{83}$$

Noting that the inner expansion $\hat{\sigma}$ vanishes as $\tau \to 0$, matching with σ requires that $a_0 = 0$. Substituting (83) into (82) and equating terms of equal powers of τ gives $A_0 = -\frac{1}{4}$, $a_1 = (2)^{1/2}$, $a_2 = \frac{1}{3}$ and $a_3 = 2(2)^{1/2}/9$. Therefore

$$\sigma_0 = (2\tau)^{1/2} + \tfrac{1}{3}\tau + \frac{2(2)^{1/2}}{9}\tau^{3/2}. \tag{84}$$

The constants A_1 and A_2 are now determined. Starting with (32a) and expanding for small σ_0 we obtain

$$\sigma_1 = -A_1\left(\frac{1}{\sigma_0}+1\right). \tag{85}$$

Taking the limit as $\tau \to 0$ and using (84) we have

$$\sigma_1 = -\frac{A_1}{(2\tau)^{1/2}}. \tag{86}$$

Working now with σ_2 in (40a) and using binomial expansions, we obtain

$$\sigma_2 = \frac{1}{(2+\sigma_0)}\left(\frac{1}{\sigma_0}+1+\frac{\sigma_0}{2}\right) + A_2\left(\frac{1}{\sigma_0}+1\right). \tag{87}$$

Taking the limit as $\tau \to 0$ and using (84), equation (87) becomes

$$\sigma_2 = (\tfrac{1}{2}+A_2)\frac{1}{(2\tau)^{1/2}}. \tag{88}$$

Substituting (84), (86) and (88) into (80) we obtain

$$\sigma = (2\tau)^{1/2} + \frac{\tau}{3} + \frac{2(2)^{1/2}}{9}\tau^{3/2} - \varepsilon^{1/2}\frac{A_1}{(2\tau)^{1/2}} + \varepsilon(\tfrac{1}{2}+A_2)\frac{1}{(2\tau)^{1/2}}. \tag{89}$$

The inner expansion $\hat{\sigma}$ is now obtained by substituting (64), (72) and (78a) into (81), taking the limit as $\mathscr{T} \to \infty$ and rewriting in outer variables.

$$\sigma = (2\tau)^{1/2} + \frac{\tau}{3} + \frac{\varepsilon^{3/2}(1-\beta)(\pi\alpha)^{1/2}}{(\tau)^{1/2}}\sum_{K=0}^{\infty}\frac{1}{\lambda_K}a_K C_K. \tag{90}$$

Matching (89) with (90) gives

$$A_1 = 0, \quad A_2 = -\tfrac{1}{2}. \tag{91}$$

We will now consider the isothermal case to determine the constants B_0 and B_1. The outer solutions for σ_0 and σ_1 are given by (23b) and (32b), respectively. The inner solutions for $\hat{\sigma}_0$, $\hat{\sigma}_1$ and $\hat{\sigma}_2$ are given by (64), (72) and (78b). The solution for σ_0 as given by (23b) is not suitable for matching. To overcome this difficulty we return to (24) and simplify it using binomial expansion for $\ln(1-\sigma_0)$ where $|(1+\phi_i)\sigma_0/\ln\beta| < 1$ to obtain

$$\mathrm{d}\tau = \sigma_0\,\mathrm{d}\sigma_0 - \left(\frac{\phi_i}{\ln\beta} + \frac{1}{2}\right)\sigma_0^2\,\mathrm{d}\sigma_0.$$

Integrating we obtain

$$\tau + B_0 = \frac{\sigma_0^2}{2} - \frac{1}{3}\left(\frac{\phi_i}{\ln\beta} + \frac{1}{2}\right)\sigma_0^3. \tag{92}$$

Equation (92) is still not suitable for matching. Following the same procedure used in the insulated case we assume that

$$\sigma_0 = b_0 + b_1(\tau)^{1/2} + b_2\tau + b_3\tau^{3/2}. \tag{93}$$

Noting that the inner expansion $\hat{\sigma}$ vanishes as $\tau \to 0$, matching with the outer expansion σ requires that $b_0 = 0$. Substituting (93) into (92) and equating terms of equal powers of τ gives $B_0 = 0$, $b_1 = (2)^{1/2}$, $b_2 = \frac{2}{3}(\phi_i/\ln\beta) + \frac{1}{3}$ and $b_3 = 5/18(2)^{1/2}(1+2\phi_i/\ln\beta)^2$. Therefore

$$\sigma_0 = (2\tau)^{1/2} + \frac{1}{3}\left(1 + \frac{2\phi_i}{\ln\beta}\right)\tau + \frac{5}{18(2)^{1/2}}\left(1 + \frac{2\phi_i}{\ln\beta}\right)^2\tau^{3/2}. \tag{94}$$

Substituting (32b) and (94) into (80) we obtain

$$\sigma = (2\tau)^{1/2} + \frac{1}{3}\left(1 + \frac{2\phi_i}{\ln\beta}\right)\tau + \frac{5}{18(2)^{1/2}}\left(1 + \frac{2\phi_i}{\ln\beta}\right)^2\tau^{3/2} + \varepsilon^{1/2}\frac{B_1}{(2\tau)^{1/2} + \frac{1}{3}\left(1 + \frac{2\phi_i}{\ln\beta}\right)\tau + \frac{5}{18(2)^{1/2}}\left(1 + \frac{2\phi_i}{\ln\beta}\right)^2\tau^{3/2}}. \tag{95}$$

The inner expansion $\hat{\sigma}$ is now obtained by substituting (64), (72), and (78b) into (81), taking the limit as $\mathscr{T} \to \infty$ and rewriting in outer variables

$$\sigma = (2\tau)^{1/2} + \frac{1}{3}\left(1 + \frac{2\phi_i}{\ln\beta}\right)\tau + \frac{\varepsilon^{3/2}}{(\tau)^{1/2}}\frac{(1-\beta)(\pi\alpha)^{1/2}}{2}\sum_{K=0}^{\infty}\frac{b_K D_K}{\gamma_K}. \tag{96}$$

Matching (95) with (96) gives

$$B_1 = 0. \tag{97}$$

Thus matching of the inner and outer expansions for the interface motion yields all the unknown constants of integration for both the insulated and isothermal cases. Having thus determined these constants, the solid-phase temperature for the inner and outer expansions must also match. This was carried out and it was established that matching of the temperature expansions is consistent with matching of the interface motion.

4. RESULTS

Of particular interest in free boundary problems is the determination of the interface motion. For the insulated case the outer expansion to order $\varepsilon^{3/2}$ is found to be independent of the parameters α, β and ϕ_i. These effects appear in the second order correction, $\hat{\sigma}_2$, of the short time solution. Examination of the zero-order long-time solution, equation (23a), shows that it is regular as $\tau \to 0$ as well as for large values of τ. Thus, the zero order solution σ_0 can be used to predict total freeze time which for $\beta = 0$ is found to be 0.25. However, the second order correction is singular at both ends of the time scale. Figure 2 shows the inner and outer expansions for the case of $\alpha = 1$, $\phi_i = 2$ and $\varepsilon = 0.05$. Because of the singularity at large τ, total freeze time cannot be predicted for all values of β. Since the maximum possible interface location is given by $\sigma_{\max} = 1-\beta$, it follows from Fig. 2 that the singularity is of no consequence for all cases of β greater than approximately 0.11. This large time singularity of the outer expansion has been treated by Riley *et al.* [7] for freezing in a cylinder of a liquid which is initially at the fusion temperature. Figure 3 compares their solution with our results. Also shown in Fig. 3 is the numerical solution of Beckett [20]. Except for the region of singularity at large τ, our solution is in excellent agreement with the results of [7] and [20].

For the isothermal case the outer expansion to order $\varepsilon^{3/2}$ is independent of α. The effect of this parameter appears in the second order correction of the short time solution. While the outer expansion becomes singular as $\tau \to 0$, it is well behaved at large τ in contrast to the insulated case. Furthermore, it converges to the exact steady state solution given by

$$\sigma(\infty) = 1 - \beta^{1/(1+\phi_i)}, \tag{98}$$

as $\tau \to \infty$. Figure 4 is a plot of the normalized interface $\sigma(\tau)/\sigma(\infty)$ for $\alpha = 1$, $\phi_i = 2$ and $\varepsilon = 0.05$. Inner and outer solutions corresponding to $\beta = 0.05$, 0.1, 0.15 and 0.5 are shown. As indicated in Fig. 4 the parameter β plays an important role in the interface solution. The breakdown in matching for $\beta \geqslant 0.15$ is traced to the condition $|(1+\phi_i)\sigma_0/\ln\beta| < 1$ used in the analytic determination of the constant B_0. As β and ϕ_i are increased this analytic approximation becomes invalid. The value $B_0 = 0$ determined by this approach is therefore an approximation. In general B_0 is a function of β and ϕ_i. One way of relaxing the limitation on this solution is to carry out the matching numerically, adjusting the value of B_0 until the outer and inner expansions match in an overlapping domain. This was

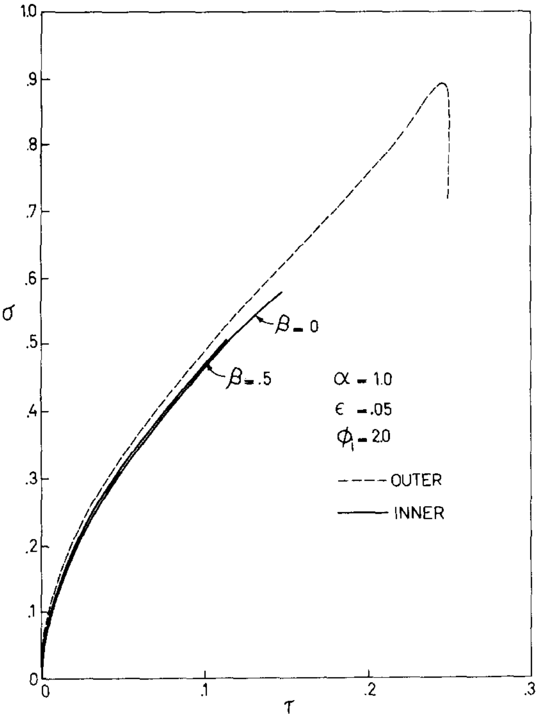

FIG. 2. Inner and outer expansions of the interface location—insulated boundary.

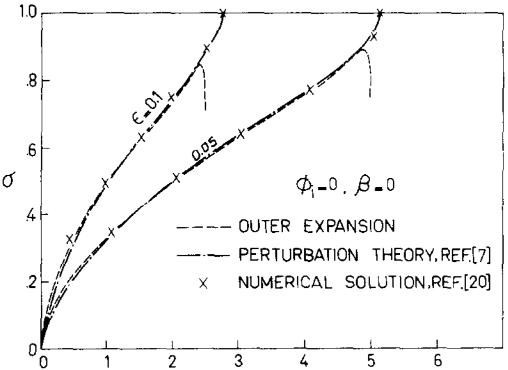

FIG. 3. Comparison of the outer expansion with the solutions of Riley *et al.* [7] and Beckett [20].

done for $\phi_i = 2$ and $\beta = 0.15$ and 0.5. It was found that a choice of $B_0 = -0.266$ for $\beta = 0.15$ and $B_0 = -0.07$ for $\beta = 0.5$ shifts the corresponding outer expansions towards the inner solutions such that the two overlap as shown in Fig. 4.

The inner and outer expansions for the liquid-phase temperature distribution are shown in Fig. 5 for the insulated case with $\alpha = 1$ and $\beta = 0.5$. The transient characteristic of the temperature profile is exhibited by the short time solution. However, the long time expansion to order $\varepsilon^{3/2}$ gives the trivial solution $\phi = 0$.

For the isothermal case a steady state solution exists which is given by the zero-order long time solution (22b) corresponding to the exact steady state interface location predicted by (98). Figure 6 gives the short and long-time solutions for $\alpha = 1$, $\beta = 0.1$, $\eta = 0.5$ and $\varepsilon = 0.05$. The long-time expansion to order ε gives the quasi-steady solution. This solution becomes invalid as $\tau \to 0$. This non-uniformity is associated with the singularity of σ_0 in equation (23b).

The long-time solid phase temperature solutions for the insulated and isothermal cases become non-uniform as $\tau \to 0$. For the insulated case the non-uniformity develops in the second-order correction θ_2 and is traced to the singularity of σ_2. For the isothermal case the non-uniformity is attributed to σ_0 and appears in the zero-order solution θ_0. Examination of the outer expansions shows that to order $\varepsilon^{3/2}$ the solutions are independent of α, β and ϕ_i. The effect of these parameters develops in the second order correction, $\hat{\theta}_2$, of the short-time solution.

5. CONCLUSION

The two cases considered, the insulated and the isothermal inner boundary conditions, are found to exhibit contrasting behavior. This contrast is traced to the physics of the problem which is fundamentally different for the two cases.

(i) For the insulated case, except on the short time scale, the effect of the initial temperature T_i is minor when the sensible heat is small compared to the latent heat of fusion ($\varepsilon \ll 1$). Consequently solutions do not depart markedly from those corresponding to liquids which are initially at the fusion temperature T_f.

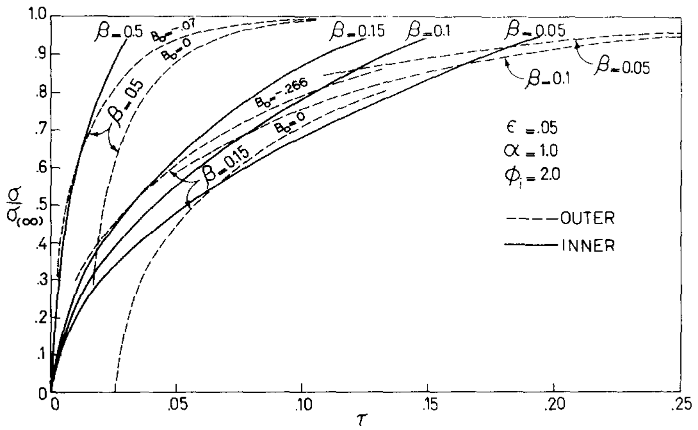

FIG. 4. Inner and outer expansions of the interface location—isothermal boundary.

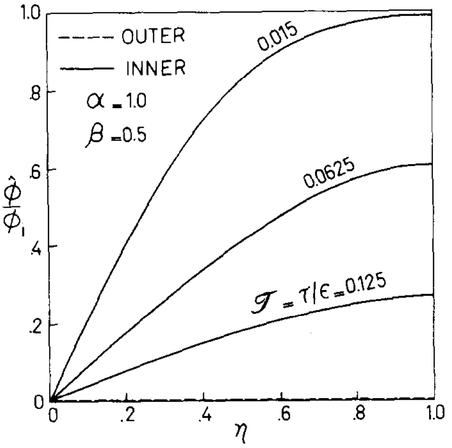

FIG. 5. Inner and outer expansions of the liquid phase temperature distribution—insulated boundary.

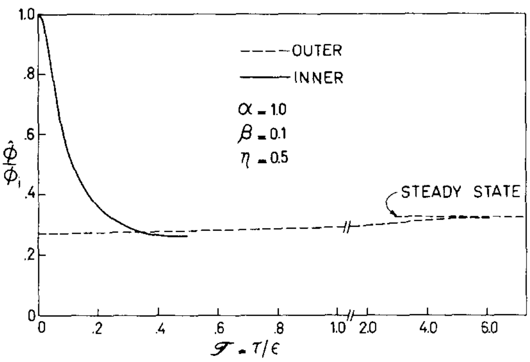

FIG. 6. Inner and outer expansions of the liquid phase temperature distribution—isothermal boundary.

(ii) For the isothermal case the initial temperature plays an important role in the solution even if the sensible heat is small compared to the latent heat of fusion. Therefore, approximate solutions which assume the initial temperature to be the same as the fusion temperature are invalid.

(iii) The effect of curvature β on the solution is minor for the insulated case but significant for the isothermal case.

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SOLUTIONS DE PERTURBATION POUR LA FUSION OU LA SOLIDIFICATION DANS DES REGIONS ANNULAIRES INITIALIMENT A UNE TEMPERATURE DIFFERENTE DE CELLE DE FUSION

Résumé—On étudie la solidification centripète d'un liquide dans un espace annulaire initialement à une température différente de celle de la fusion. La surface cylindrique externe est maintenue à une température au dessous de celle de solidification que la surface cylindrique interne est soit isolée, soit à température constante. On obtient de nouvelles solutions de perturbation pour la distribution de température et le déplacement de l'interface. Le paramètre de perturbation

$$\varepsilon = \frac{C^s(T^f - T^a)}{\mathcal{L}}$$

est le rapport de la chaleur sensible de la phase solide à la chaleur latente de fusion. La non-uniformité des solutions à longue échelle de temps est traitée en construisant des développements internes à petite échelle de temps. Les deux solutions sont éprouvées en utilisant la théorie asymptotique. Les solutions pour le cas d'isolation ne se démarquent pas fortement, excepté pour les temps courts, des solutions correspondantes pour des liquides initialement à la température de fusion T^f. Par contre, les solutions relatives à la limite interne isotherme diffèrent nettement de celles obtenues pour une température initiale égale à T^f. Ceci est vrai même si la chaleur sensible est petite par rapport à la chaleur latente de fusion. La courbure joue un rôle mineur dans le déplacement de l'interface dans le cas de l'isolation tandis que son effet est dramatique dans le cas isotherme.

STÖRUNGSLÖSUNGEN FÜR SCHMELZEN ODER GEFRIEREN IN KREISRINGGEBIETEN, DIE SICH ANFANGS NICHT AUF SCHMELZTEMPERATUR BEFINDEN

Zusammenfassung—Die Arbeit behandelt den inneren Erstarrungsvorgang von Flüssigkeit in einem Kreisringgebiet, das anfangs nicht die Schmelztemperatur hat. Die äußere Zylinderoberfläche wird unterhalb der Erstarrungstemperatur gehalten, während der innere zylindrische Rand entweder als isoliert oder auf konstanter Temperatur befindlich angenommen wird. Für die Temperaturverteilung und die Bewegung der Phasengrenze wurden neue Störungslösungen gefunden. Der Störungsparameter

$$E = \frac{c_s(T_f - T_a)}{L}$$

ist das Verhältnis der fühlbaren Wärme der festen Phase zur latenten Schmelzwärme. Die Ungleichförmigkeit der Langzeit-Lösungen wird durch Reihenentwicklung im Kurzzeit-Bereich behoben. Die beiden Lösungen werden mittels der Asymtoten-Theorie ineinander überführt. Die Lösungen für den isolierten Fall unterscheiden sich nicht deutlich—ausgenommen im Kurzzeit-Bereich—von den entsprechenden Lösungen für Flüssigkeiten, die sich anfangs auf der Schmelztemperatur T_f befinden. Im Gegensatz dazu unterscheiden sich die Lösungen für die isotherme innere Berandung wesentlich von denen mit einer Anfangstemperatur von T_f. Das trifft sogar noch zu, wenn die fühlbare Wärme klein im Vergleich zur latenten Schmelzwärme ist. In ähnlicher Weise spielt die Krümmung bei der Bewegung der Phasengrenze im isolierten Fall nur eine untergeordnete Rolle, während ihr Einfluß im isothermen Fall äußerst stark ist.

РЕШЕНИЕ МЕТОДОМ ВОЗМУЩЕНИЙ ЗАДАЧИ О ПЛАВЛЕНИИ ИЛИ ЗАТВЕРДЕВАНИИ В КОЛЬЦЕВЫХ ЗОНАХ С НАЧАЛЬНОЙ ТЕМПЕРАТУРОЙ, ОТЛИЧНОЙ ОТ ТЕМПЕРАТУРЫ ПЛАВЛЕНИЯ

Аннотация — Рассматривается затвердевание жидкости в кольцевом пространстве с начальной температурой, отличной от температуры плавления, начинающееся во внешней области. Внешняя поверхность цилиндра поддерживается при температуре ниже температуры замерзания. Предполагается, что внутренняя поверхность либо изолирована, либо поддерживается при постоянной температуре. Методом возмущений получены распределение температуры и закон движения границы раздела. Параметром возмущений,

$$\varepsilon = \frac{c_s(T_f - T_a)}{L},$$

является отношение теплосодержания твердой фазы к скрытой теплоте плавления. Неоднородность решений по длинновременной шкале рассматривается с помощью внутреннего разложения по коротковременной шкале. Эти решения сшиваются с помощью асимптотической теории. В случае изолированной внутренней поверхности в решениях не наблюдается заметных отклонений (за исключением решений по коротковременной шкале) от соответствующих решений для жидкостей с начальной температурой, равной температуре плавления T_f. Решения же для изотермической внутренней границы значительно отличаются от соответствующих решений с начальной температурой, равной температуре плавления T_f. Это расхождение наблюдается даже в том случае, когда теплосодержание невелико по сравнению со скрытой теплотой плавления. Аналогичным образом, кривизна оказывает незначительное влияние на перемещение границы раздела фаз в случае изолированной внутренней поверхности, тогда как в случае изотермической поверхности её влияние весьма существенно.